5-2007

Unit Root Log Periodogram Regression

Peter C. B. PHILLIPS
Singapore Management University, peterphillips@smu.edu.sg

DOI: https://doi.org/10.1016/j.jeconom.2006.05.017

Published in Journal of Econometrics, Volume 138, Issue 1, May 2007, Pages 104-124
https://doi.org/10.1016/j.jeconom.2006.05.017

Unit root log periodogram regression

Peter C.B. Phillips[a,b,c,*]

[a]*Cowles Foundation for Research in Economics, Yale University, Box 208281 Yale Station, New Haven, CT 06520, USA*
[b]*University of York, UK*
[c]*University of Auckland, New Zealand*

Abstract

Log periodogram (LP) regression is shown to be consistent and to have a mixed normal limit distribution when the memory parameter $d = 1$. Gaussian errors are not required. The proof relies on a new result showing that asymptotically infinite collections of discrete Fourier transforms (dft's) of a short memory process at the fundamental frequencies in the vicinity of the origin can be treated as asymptotically independent normal variates, provided one does not include too many dft's in the collection.

JEL classification: C22

Keywords: Asymptotic independence; Discrete Fourier transform; Fractional integration; Log periodogram regression; Long memory parameter; Nonstationarity; Semiparametric estimation; Unit root

1. Introduction

For the last two decades a primary focus in econometric research has been on the long-run properties of economic time series, including the intrinsic memory properties displayed by individual series and the existence of long-run relationships between series. Many economic time series, such as inflation and interest rates, display long memory in the sense that their temporal autocorrelations decay slowly (if at all) and characterizing this property

*Corresponding author at: Cowles Foundation for Research in Economics, Yale University, Box 208281 Yale Station, New Haven, CT 06520, USA. Tel.: +1 203 421 4708; fax: +1 203 432 5429.
E-mail address: peter.phillips@yale.edu.

empirically has presented many econometric challenges. A particularly attractive econometric approach is semiparametric, in which the parameter (d) that characterizes memory in the data is estimated without making any delimiting assumptions about the short memory components in the data generating process. Accordingly, semiparametric estimation of the parameter d in fractionally integrated ($I(d)$) time series has received much recent study.

In applied work, $I(d)$ processes with fractional $d>0$ have been found to provide good empirical models for financial time series and volatility measures, as well as certain macroeconomic time series like inflation, money stock, and interest rates. Robinson (1994a) and Baillie (1996) reviewed aspects of this work relevant to econometrics up to the mid 1990s and there has been much work in the field since then. Growing evidence in applied work indicates that fractionally integrated processes can describe certain long range characteristics of economic data rather well, including the volatility of financial asset returns, forward exchange market premia, interest rate differentials, and inflation rates.

Two commonly used semiparametric estimators are log periodogram (LP) regression (Geweke and Porter-Hudak, 1983) and local Whittle (LW) estimation (Künsch, 1986). LW estimation involves numerical optimization of the LW likelihood and is attractive because it is asymptotically efficient. LP regression is popular because of its convenience, which stems from the simplicity of its construction as a linear regression estimator, and it has been extensively used in applied econometric research.

Let X_t be a fractional process satisfying

$$(1-L)^d X_t = u_t, \quad t \geqslant 0, \;\; X_0 = u_0 = 0, \tag{1}$$

where u_t is stationary with zero mean, finite moments to order p and continuous spectral density $f_u(\lambda)>0$. The parameter d in (1) measures the extent of the memory or long range dependence in X_t. The present paper concentrates on the special case where $d=1$, which corresponds to the important unit root case.

The LP estimator $\widehat{d}$ is obtained from the least squares regression

$$\log(I_X(\lambda_s)) = \widehat{c} - \widehat{d}\log|1-\mathrm{e}^{\mathrm{i}\lambda_s}|^2 + \text{residual} \tag{2}$$

taken over fundamental frequencies $\{\lambda_s = 2\pi s/n : s=1,\ldots,m\}$ for some $m<n$. Setting $a_s = \log|1-\mathrm{e}^{\mathrm{i}\lambda_s}|$ and $x_s = a_s - \overline{a}$, where $\overline{a} = m^{-1}\sum_{s=1}^m a_s$, we have

$$\widehat{d} = -\frac{1}{2}\frac{\sum_{s=1}^m x_s \log I_X(\lambda_s)}{\sum_{s=1}^m x_s^2}, \tag{3}$$

where $I_X(\lambda_s) = w_X(\lambda_s)w_X(\lambda_s)^*$ is the periodogram and $w_X(\lambda_s)$ is the discrete Fourier transform (dft), $w_X(\lambda_s) = (1/\sqrt{2\pi n})\sum_{t=1}^n X_t \mathrm{e}^{\mathrm{i}t\lambda_s}$ of the time series X_t. The regression (2) is motivated by the form of the log spectrum of X_t and has appeal because of its nonparametric treatment of u_t and the convenience of linear least squares. Under Gaussian assumptions and in the stationary case, where $d \in \left(-\frac{1}{2},\frac{1}{2}\right)$, Robinson (1995) developed consistency and asymptotic normality results for a version of $\widehat{d}$ which trims out low frequencies periodogram ordinates (i.e. takes $s \geqslant l$, for some $m>l>1$), as suggested by Künsch (1986). Hurvich and Beltrao (1993) have developed data-driven criteria for the selection of m; and Hurvich et al. (1998) extend Robinson's (1995) results to include low frequencies ordinates and find an optimal choice of the number of periodogram ordinates in the regression. These papers provide a foundation of asymptotic theory validating (2) for use with Gaussian data in the stationary case.

The present paper develops an asymptotic theory for LP regression in the unit root case. Gaussianity is not assumed and it is shown that, when $d = 1$, $\widehat{d}$ has a mixed normal limit distribution whose variance is smaller than the variance, $\pi^2/24m$, that is known to apply when $d \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. While highly specialized, the unit root case is of interest for several reasons. First, the unit root model has received a vast amount of attention in the literature on nonstationary autoregression, but is presently not covered by the existing theory of semiparametric estimation of d. The case is particularly important in economic applications. Second, it is well known that the corresponding semiparametric unit root limit theory is nonstandard (Phillips, 1987) and it is of interest to discover whether there are any unusual features to the limit theory in the semiparametric estimation of d when $d = 1$. Third, it is known (Kim and Phillips, 1999) that $\widehat{d}$ is inconsistent when $d > 1$ with $\widehat{d} \to_p 1$, but consistent for $\frac{1}{2} < d < 1$. So $d = 1$ turns out to be the boundary case for consistent estimation by LP regression. Similar consistency, inconsistency and boundary results apply to the LW estimator of d, as has been shown in Phillips and Shimotsu (2004). It would appear, therefore, that the unit root case has some special characteristics in the fractional domain as well as those which are already well known in the autoregressive domain.

The work reported here complements other recent research on LP regression and testing in nonstationary and non-Gaussian cases. Hurvich and Ray (1995) looked at the behavior of periodogram ordinates of a fractionally integrated process with memory parameter $d \in [0.5, 1.5)$, found evidence of bias in LP regression when $d > 1$, and evidence of asymptotic unbiasedness under Gaussianity when $d = 1$. Velasco (1999a) showed consistency (under Gaussian assumptions and for $\frac{1}{2} < d < 1$) of an LP estimator that trims out low frequency ordinates, gave a CLT for $d \in \left[\frac{1}{2}, \frac{3}{4}\right]$, and gave a CLT for trimmed and tapered estimates when $d > \frac{3}{4}$. Velasco (2000) showed consistency of an LP regression estimator under non-Gaussian u_t when $d \in (0, \frac{1}{2})$ and periodogram ordinates are pooled and frequencies are trimmed from the origin. Kim and Phillips (1999) showed LP regression is consistent for $\frac{1}{2} < d < 1$ without requiring Gaussianity or trimming, as well as demonstrating inconsistency for $d > 1$. Some simulation results covering nonstationary cases were reported in Hurvich and Ray (1995) and Velasco (1999b), both revealing evidence of estimation bias when $d > 1$. Finally, Robinson (1994b) and Tanaka (1999) show how to use Lagrange multiplier and Wald theory for testing values of d in parametric models that include both stationary and nonstationary cases, but exclude weak nonparametric dependence.

The limit theory is given in Section 3 and Section 2 provides some preliminary theory and assumptions. The mixed normal limit for $\widehat{d}$ is derived using a conditioning argument and embedding that appear to be useful outside the present context. In particular, the embedding reveals that, in the non-Gaussian case, one can uniformly approximate an asymptotically infinite collection of dft's of a short memory time series like u_t at fundamental frequencies in the vicinity of the origin by a sequence of independent, identically distributed complex normal variates, provided the number of frequency ordinates $m = o(n^{1/2-1/p})$. This result is of independent interest and should have applications in frequency domain asymptotics beyond those of LP regression. Some conclusions are given in Section 4 and proofs and other technical material are given in Section 5.

2. Preliminaries

The fractionally integrated process X_t is defined as in (1), with $u_j = 0$ for all $j \leqslant 0$. Explicit conditions on u_t $(t > 0)$ are given in the following.

Assumption L. For all $t > 0$, u_t has Wold representation

$$u_t = C(L)\varepsilon_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}, \quad \sum_{j=0}^{\infty} j|c_j| < \infty, \quad C(1) \neq 0, \tag{4}$$

with $\varepsilon_t = \text{iid}(0, \sigma_\varepsilon^2)$ with $\mathrm{E}(|\varepsilon_t|^p) < \infty$, for some $p > 2$.

The specificity of (4) involves a loss of generality in comparison with purely local to zero assumptions about the short memory spectrum that are used in other work (e.g. Robinson, 1995, and Velasco, 1999a) and which are appealing in view of the semiparametric nature of the model. Nevertheless, the summability condition in (4) is satisfied by a wide class of parametric and nonparametric models for u_t and, in conjunction with the moment condition, enables the use of the techniques in Phillips and Solo (1992) and embedding arguments for the partial sums of u_t, both of which simplify our approach. The spectral density of u_t is $f_u(\lambda) = (\sigma_\varepsilon^2/2\pi)|\sum_{j=0}^{\infty} c_j \mathrm{e}^{\mathrm{i}j\lambda}|^2$.

From Eq. (16) of Phillips (1999), we have the decomposition (taking $X_0 = 0$ here and in what follows to simplify formulae with no essential loss of generality)

$$w_X(\lambda) = \frac{w_u(\lambda)}{1 - \mathrm{e}^{\mathrm{i}\lambda}} - \frac{\mathrm{e}^{\mathrm{i}(n+1)\lambda}}{1 - \mathrm{e}^{\mathrm{i}\lambda}} \frac{X_n}{\sqrt{2\pi n}} = \frac{w_u(\lambda)}{1 - \mathrm{e}^{\mathrm{i}\lambda}} - \frac{\mathrm{e}^{\mathrm{i}(n+1)\lambda}}{1 - \mathrm{e}^{\mathrm{i}\lambda}} w_u(\lambda_0), \tag{5}$$

a result that may be obtained directly (e.g. by partial summation—see (22) below). It is apparent that both components of (5) influence the asymptotic behavior of the data dft $w_X(\lambda)$ when $d = 1$. In particular, when evaluated at the fundamental frequencies $\lambda_s = 2\pi s/n$, $s = 1, \ldots, n$, the component $(2\pi n)^{-1/2} X_n$ in (5) does not depend on λ_s and yet this term influences the asymptotic behavior of $w_X(\lambda_s)$ for all values of s. This property means that $w_X(\lambda_s)$ is spatially correlated across all the fundamental frequencies. In effect, there is leakage across all the fundamental frequencies from the zero frequency ($s = 0$), i.e. from $w_u(\lambda_0) = (2\pi n)^{-1/2} X_n$.

A further complicating factor in LP regression is that one needs to work with a logarithmic function of the periodogram ordinates. In effect, the model underlying the empirical regression (2) involves the logarithm of the squared modulus of (5) at λ_s, i.e.

$$\log |w_X(\lambda_s)|^2 = -2 \log |1 - \mathrm{e}^{\mathrm{i}\lambda_s}| + \log \left| w_u(\lambda_s) - \mathrm{e}^{\mathrm{i}\lambda_s} \frac{X_n}{\sqrt{2\pi n}} \right|^2. \tag{6}$$

The second term in (6) involves the dft of the stationary errors, $w_u(\lambda_s)$, coupled with the leakage factor $w_u(\lambda_0) = (2\pi n)^{-1/2} X_n$ from $s = 0$. The asymptotic analysis of LP regression requires that we treat both the nonlinearity in (6) and the spatial correlation arising from this leakage.

3. LP regression in the unit root case

The approach we take to deal with the complications just mentioned is to use an embedding and conditioning argument. First, we expand the probability space as needed so that the processes are defined in such a way that the data can be represented up to a negligible error in terms of a Brownian motion defined on the same space. Since our interest is in a limit distribution theory it is, in fact, sufficient that the representations hold in probability rather than almost surely. A result of this type was given in Akonom (1993, Theorem 3) and is included in the following lemma in the form (7), for which we supply a simple proof that uses the device in Phillips and Solo (1992).

Lemma 3.1. *Let $S_k = \sum_{j=1}^{k} u_j$ for $k \geqslant 1$, and $S_0 = 0$, for $k = 0$, where u_j satisfies Assumption L. Then, the probability space on which the u_j and S_k are defined can be expanded in such a way that there is a process distributionally equivalent to S_k and a Brownian motion $B(\cdot)$ with variance $\sigma^2 = 2\pi f_u(0)$ on the new space for which*

$$\sup_{0 \leqslant k \leqslant n} \left| \frac{S_k}{\sqrt{n}} - B\left(\frac{k}{n}\right) \right| = o_p\left(\frac{1}{n^{1/2-1/p}}\right). \tag{7}$$

If $\mathrm{E}|\varepsilon_t|^q < \infty$ for $q > 2p > 4$, then (7) holds almost surely.

In Lemma 3.1 it is assumed that the maximal moment exponents p and q are finite. When $p = \infty$, recent results on (multivariate) strong approximation (e.g. Zaitsev, 1998) for partial sums of iid (vector) variates can be used in combination with the Phillips–Solo (1992) device to prove a strong approximation for partial sums of a (multivariate) linear process. These results ensure a uniform error of $O_{\text{a.s.}}(\log n/\sqrt{n})$ when the component variates have exponential moments. Under such conditions, it is to be expected that the results derived below may be correspondingly strengthened. That case is not analyzed here but we do provide results for the Gaussian case.

With approximation (7) in hand, we establish the following useful stochastic integral representation for the asymptotically infinite family of dft's $\{w_u(\lambda_s) : s = 0, \ldots, m\}$.

Theorem 3.2. *If u_t satisfies Assumption L and if $m/n^{1/2-1/p} \to 0$, as $n \to \infty$, then the family of dft's $\{w_u(\lambda_s) : s = 0, \ldots, m\}$ may be asymptotically represented as $n \to \infty$ in the form of a sequence of independent normal random variables ξ_s such that*

$$w_u(\lambda_s) = \xi_s + o_p\left(\frac{m}{n^{1/2-1/p}}\right), \tag{8}$$

uniformly over $s \leqslant m$, where $\xi_s = (1/\sqrt{2\pi}) \int_0^1 e^{2\pi i s r}\, dB(r)$ and the collection $\{\xi_s\}_{s=1}^{m}$ are iid complex $N(0, \sigma^2)$ and independent of the real normal variate $\eta = \xi_0 = (1/\sqrt{2\pi}) B(1)$.

It follows from Theorem 3.2 that if $m \to \infty$ and $m = o(n^{1/2-1/p})$, then the asymptotically infinite collection of dft's $\{w_u(\lambda_s) : s = 0, \ldots, m\}$ form a set of asymptotically independent variates. Under a variety of conditions, analogous results have been shown by many authors (e.g. Hannan, 1973) but apparently only for a finite collection of dft's. Thus, it is well known that a finite collection of dft's, such as $\{w_u(\lambda_s) : s = 0, \ldots, M\}$ for M fixed, form an asymptotically independent set of variates. On the other hand, it is also known (e.g. Fay and Soulier, 1999) that in many applications involving the full set of variates $\{w_u(\lambda_s) : s = 0, \ldots, n-1\}$ one cannot treat the dft's simultaneously as if they were a sequence of asymptotically independent normal random variables except in the case where the underlying components u_t are themselves normally distributed. Theorem 3.2 shows that an intermediate result is available and that asymptotically infinite collections of dft's can be treated as asymptotically independent normal variates, provided one does not include too many in the collection. Thus, Theorem 3.2 can be expected to be relevant in frequency domain applications (like LP regression) that involve local averaging of dft's or functionals of dft's like the periodogram. Theorem 3.2 is given for collections of dft's in the vicinity in the origin as that is all we need for the LP regression application. However, with some modifications to the proof of Theorem 3.2, a related result can be proved for asymptotically infinite collections of dft's in the vicinity of an arbitrary fixed frequency. This work will be reported elsewhere, as it is not directly related to the subject of the present application.

The simple form of (8) enables us to develop a limit theory for frequency averages of the second term in (6). LP asymptotics follow directly. The outcome is the following result, which gives the asymptotic distribution of $\widehat{d}$ when $d = 1$.

Theorem 3.3. *Let X_t follow* (1) *with $d = 1$ and u_t satisfy Assumption L. If*

$$\frac{m^2 \log m}{n^{1/2-1/p}} + \frac{1}{m} \to 0, \tag{9}$$

then

$$\sqrt{m}(\widehat{d} - d) \to_d MN\left(0, \frac{1}{4}\sigma^2(W)\right) \equiv \int_0^\infty \mathrm{N}\left(0, \frac{1}{4}\sigma^2(w)\right) \mathrm{pdf}(w)\, \mathrm{d}w. \tag{10}$$

Here, W is χ_1^2 with $\mathrm{pdf}(w) = \left[2^{1/2}\Gamma\left(\frac{1}{2}\right)\right]^{-1} \mathrm{e}^{-w/2} w^{-1/2}$, *and* $\sigma^2(w) = \mathrm{Var}\{\log(\chi_2^2(2w))\}$, *where $\chi_2^2(\delta)$ denotes a χ_2^2 variate with noncentrality parameter δ. The explicit form of $\sigma^2(w)$ is*

$$\sigma^2(w) = \mathrm{e}^{-w} \sum_{j=0}^{\infty} \frac{w^j}{j!} \{\psi(1+j)^2 + \psi'(1+j)\} - \left[\mathrm{e}^{-w} \sum_{j=0}^{\infty} \frac{w^j}{j!} \psi(1+j)\right]^2, \tag{11}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the psi function, the logarithmic derivative of the gamma function Γ, and $\psi'(z)$ is the trigamma function, its first derivative (*explicit representations of $\psi(z)$ and $\psi'(z)$ are given in* (45) *and* (46) *in the Appendix*).

If u_t is Gaussian then (10) *holds under*

$$\frac{m \log m}{n^{1/2}} + \frac{1}{m} \to 0. \tag{12}$$

Corollary 3.4. *Under the conditions of Theorem* 3.3 *the limiting variance of $\sqrt{m}(\widehat{d} - d)$ is*

$$\begin{aligned}\sigma_d^2 = {} & \frac{\pi^2}{24} + \frac{1}{4\sqrt{3}} \sum_{j=0}^{\infty} \frac{[\psi(1+j)^2 - \sum_{k=1}^{j}(1/k^2)](1/2)_j (2/3)^j}{j!} \\ & - \frac{1}{4\sqrt{5}} \sum_{j=0}^{\infty} \frac{\psi(1+j)(1/2)_j (2/5)^j}{j!} \sum_{k=0}^{\infty} \frac{\psi(1+k)}{k!} \left(\frac{1}{2}+j\right)_k \left(\frac{2}{5}\right)^k,\end{aligned} \tag{13}$$

where $(a)_j = \Gamma(a+j)/\Gamma(j)$ is the forward factorial function.

Remark 3.5. (a) Condition (9) on the frequency band $\{\lambda_s,\ 1 < s < m\}$ holds when $m = \mathrm{O}(n^{1/4-(1/2p)-\delta})$ for any $\delta > 0$, which is far more restrictive than the $\mathrm{o}(n^{4/5})$ rate of Hurvich et al. (1998, Theorem 2) which was established under Gaussian errors, and even the $\mathrm{O}(n^{1/2})$ rate suggested in Geweke and Porter-Hudak (1983), both of which refer to the stationary case. The highly restrictive nature of (9) is most likely an artifact of our simple method of proof, which relies first on the representation (8), which places a preliminary restriction on m, and, second, on a bound, (31), on the reciprocals of certain random variables that we use in the treatment of the logarithmic function in the proof of (10) and which leads to a further restriction on m. It seems likely that the latter can be relaxed, although we have not attempted to do so. Under Gaussianity, the weaker condition (12) is sufficient, which requires $m = \mathrm{O}(n^{1/2-\delta})$ for $\delta > 0$.

(b) Numerical evaluation of (13) gives $\sigma_d^2 = 0.3948$, which is slightly smaller than $\pi^2/24 = 0.4112$, the limiting variance of $\sqrt{m}(\widehat{d} - d)$ in the stationary case $|d| < \frac{1}{2}$. Thus, the limit distribution of the LP estimator in the unit root case, although mixed normal, has smaller variance than in the stationary case, leaving aside issues of rates of convergence attendant to the allowable expansion rate of m.

(c) Phillips and Shimotsu (2004) established that the LW $\tilde{d}$ estimator of d also has a mixed normal limit distribution when $d = 1$. They calculated the variance of the limit distribution of $\sqrt{m}(\tilde{d} - d)$ to be 0.2028, which is therefore less than the limiting variance of the LP estimator when $d = 1$ and less than the variance of the LW estimator in the stationary case (0.25).

4. Conclusion

The above results help to complete the limit theory for the LP regression estimator, but they are not immediately useful in estimation or inference in view of the inconsistency of LP regression for $d > 1$. Consistent semiparametric estimation for all values of d by LP regression without trimming or tapering may be achieved using an exact LP (ELP) regression procedure, as discussed in Phillips (1999), although a rigorous asymptotic theory for this estimator is yet to be developed. However, this approach is analogous to the exact local Whittle (ELW) estimator of Phillips (1999) and Shimotsu and Phillips (2005). Both estimators involve non-linear optimization, and thus the ELP procedure loses the advantage of linear regression that makes LP appealing in practice. Moreover, since ELW is the more efficient asymptotically, this estimator seems likely to be the preferred choice.

In addition to long memory modeling, a related focus of recent research has been models with autoregressive roots near unity, or nearly integrated processes. These models, which have roots that are within $\mathrm{O}(n^{-1})$ of unity, were studied by Chan and Wei (1987) and Phillips (1987) and the ideas therein have been extensively used in the analysis of the local behavior of unit root tests and the development of local point optimal testing procedures. Most recently, models with mildly nonstationary or mildly explosive behavior have been analyzed by Phillips and Magdalinos (2006a, b) and Giraitis and Phillips (2006). In these models, the roots deviate from unity by the sense that the deviations are $\mathrm{O}(k_n^{-1})$ where $k_n/n \to 0$ as $n \to \infty$. Such models provide a more effective bridge to the asymptotic theory that obtains in short memory stationary models and purely explosive models. There is considerable potential for extending these ideas to multivariate systems and the study of long-run relationships between series that are mildly nonstationary or even mildly explosive.

Acknowledgements

Thanks go to referees and an associate editor of an earlier version, Herman van Dijk, Chang Sik Kim and Katsumi Shimotsu for helpful comments and to the NSF for research support under Grant nos. SBR 97-30295 and SES 04-142254. Two referees, in particular, suggested corrections and alternative proofs which have materially improved the paper,

including Lemma C, which generalizes a result in an earlier version of the paper. The original version was written and circulated in March, 1999.

Appendix

Lemma A reports moment expressions which can be obtained straightforwardly from known results (e.g. Gradshteyn and Ryzhik, 1994—hereafter GR—and Johnson et al., 1995) and is given here for convenience. Lemma B gives the limit distribution of a sample average of nonlinear functions of correlated Gaussian variates. Lemma C is a technical result on the probability limit of weighted sums of reciprocals of independent variates. Both B and C are used in the proof of Theorem 3.3.

Lemma A.

$$\mathrm{E}(\log(\chi_v^2(\delta))) = \mathrm{e}^{-\delta/2}\sum_{j=0}^{\infty}\frac{(\delta/2)^j}{j!}\,\psi\left(\frac{v}{2}+j\right)+\log 2,$$

$$\mathrm{Var}(\log(\chi_v^2(\delta))) = \mathrm{e}^{-\delta/2}\sum_{j=0}^{\infty}\frac{(\delta/2)^j}{j!}\left\{\psi\left(\frac{v}{2}+j\right)^2+\psi'\left(\frac{v}{2}+j\right)\right\} - \left[\mathrm{e}^{-\delta/2}\sum_{j=0}^{\infty}\frac{(\delta/2)^j}{j!}\,\psi\left(\frac{v}{2}+j\right)\right]^2,$$

where $\chi_v^2(\delta)$ is noncentral χ^2 with v degrees of freedom and noncentrality parameter δ.

Lemma B. *Let $\xi_s = (1/\sqrt{2\pi})\int_0^1 \mathrm{e}^{2\pi\mathrm{i}sr}\,\mathrm{d}B(r)$ and $\eta = (1/\sqrt{2\pi})\int_0^1 \mathrm{d}B(r)$, where B is Brownian motion with variance $2\pi f_u(0)$. Then*

$$\frac{1}{\sqrt{m}}\sum_{s=1}^{m} x_s \log|\xi_s-\eta|^2 \to_d MN(0,\sigma^2(W)) \equiv \int_0^{\infty}\mathrm{N}(0,\sigma^2(w))\mathrm{pdf}(w)\,\mathrm{d}w,$$

where

$$\sigma^2(w) = \mathrm{e}^{-w}\sum_{j=0}^{\infty}\frac{w^j}{j!}\{\psi(1+j)^2+\psi'(1+j)\} - \left[\mathrm{e}^{-w}\sum_{j=0}^{\infty}\frac{w^j}{j!}\psi(1+j)\right]^2,$$

and W is χ_1^2.

Proof of Lemma B. Set

$$\xi_s = \frac{1}{\sqrt{2\pi}}\int_0^1 \mathrm{e}^{2\pi\mathrm{i}sr}\,\mathrm{d}B(r) = \sqrt{f_u(0)}\int_0^1 \mathrm{e}^{2\pi\mathrm{i}sr}\,\mathrm{d}W(r) = \sqrt{f_u(0)}Z_s, \quad \text{say},$$

and

$$\eta = \sqrt{f_u(0)}\,Y,$$

where $W(r)$ is standard Brownian motion, and $\{Z_s\}_1^m \equiv$ iid $\mathrm{N_c}(0,1)$ and is independent of Y, which is $\mathrm{N}(0,1)$. It is convenient to write $Z_s = \zeta_{1s}+\zeta_{2s}\mathrm{i}$. The components ζ_{1s}, ζ_{2s} in this decomposition are independent and each is $\mathrm{N}\left(0, \frac{1}{2}\right)$. Then

$$
\begin{aligned}
\log|\xi_s - \eta|^2 &= \log(f_u(0)) + \log[(\zeta_{1s} - Y)^2 + \zeta_{2s}^2] \\
&= \log(\tfrac{1}{2} f_u(0)) + \log[2\{(\zeta_{1s} - Y)^2 + \zeta_{2s}^2\}] \\
&= \log(\tfrac{1}{2} f_u(0)) + \log[G_{sY}], \quad \text{say.}
\end{aligned} \tag{14}
$$

Conditional on Y, $\zeta_{1s} - Y$ is $\mathrm{N}(-Y, \frac{1}{2})$, and so, conditional on Y,

$$
G_{sY} = \frac{(\zeta_{1s} - Y)^2 + \zeta_{2s}^2}{1/2} = 2\{(\zeta_{1s} - Y)^2 + \zeta_{2s}^2\} \equiv \chi_2^2(\delta).
$$

Thus, conditional on Y, the family $\{G_{sY}\}_1^m$ are independent and identically distributed noncentral chi-squared variates with two degrees of freedom and noncentrality parameter δ where

$$
\delta = \left(\frac{-Y}{1/\sqrt{2}}\right)^2 = 2Y^2 = 2W.
$$

It follows from Lemma A and (46) that

$$
\mathrm{E}(\log G_{sY} | Y) = \mathrm{E}(\log(\chi_2^2(2W)) | Y) := \mu_Y,
$$

and

$$
\mathrm{Var}(\log G_{sY} | Y) = \mathrm{Var}((\log(\chi_2^2(2W)) | Y)) := \sigma_Y^2.
$$

Thus, conditional on Y, $\log G_{sY}$ is iid (μ_Y, σ_Y^2). It follows that, conditional on Y,

$$
\frac{1}{\sqrt{m}} \sum_{j=1}^{m} x_s \log G_{sY} = \frac{1}{\sqrt{m}} \sum_{j=1}^{m} x_s (\log G_{sY} - \mu_Y)
$$

satisfies the Lindeberg–Feller central limit theorem (cf. Robinson, 1995, p. 1070) and we have

$$
\begin{aligned}
\left.\frac{1}{\sqrt{m}} \sum_{j=1}^{m} x_s \log G_{sY}\right|_Y \xrightarrow{d} \mathrm{N}\left(0, \sigma_Y^2 \left(\lim_{m\to\infty} \frac{1}{m} \sum_{j=1}^{m} x_s^2\right)\right) &= \mathrm{N}\left(0, \sigma_Y^2 \left(\lim_{m\to\infty} \frac{1}{m} \sum_{j=1}^{m} x_s^2\right)\right) \\
&= \mathrm{N}(0, \sigma_Y^2).
\end{aligned}
$$

The variance expression follows since, as shown in (36), $x_s = \log s/m - (1/m)\sum_{s=1}^{m} \log(s/m) + \mathrm{O}(m^2/n^2)$ and then

$$
\lim_{m\to\infty} \frac{1}{m} \sum_{j=1}^{m} x_s^2 = \int_0^1 (\log x)^2 \,\mathrm{d}x - \left(\int_0^1 (\log x)\,\mathrm{d}x\right)^2 = 1.
$$

Unconditionally, we therefore have

$$
\frac{1}{\sqrt{m}} \sum_{j=1}^{m} x_s \log G_{sY} \xrightarrow{d} MN(0, \sigma_Y^2) = \int_0^\infty \mathrm{N}(0, \sigma^2(w)) \mathrm{pdf}(w)\,\mathrm{d}w, \tag{15}
$$

where

$$\sigma^2(w) = \mathrm{e}^{-w}\sum_{j=0}^{\infty}\frac{w^j}{j!}\{\psi(1+j)^2+\psi'(1+j)\} - \left[\mathrm{e}^{-w}\sum_{j=0}^{\infty}\frac{w^j}{j!}\psi(1+j)\right]^2,$$

and $W = Y^2 = \chi_1^2$.

It follows from (14) and (15) that

$$\frac{1}{\sqrt{m}}\sum_{j=1}^{m} x_s \log|\xi_s-\eta|^2 = \frac{1}{\sqrt{m}}\sum_{j=1}^{m} x_s \log G_{sY} \xrightarrow{d} \int_0^{\infty} \mathrm{N}(0,\sigma^2(w))\mathrm{pdf}(w)\,\mathrm{d}w,$$

as stated. $\square$

The following lemma was suggested by a referee and generalizes a similar lemma given in an earlier version of this paper for Gaussian sequences:

Lemma C. *Let $\{\zeta_s : s = 1,\ldots,m\}$ be an iid sequence with probability density $f(\cdot)$ that satisfies*

(A1) *f is symmetric about the origin and both f and f' are continuous.*
(A2) *As $x\to\infty$, $f(x)\log x = \mathrm{o}(1)$ and $x^2 f'(x)\log x = \mathrm{O}(1)$.*
Let $\{x_{ms} : m\geqslant 1; s = 1,\ldots,m\}$ be a series of real sequences for which
(A3) $(1/m)\sum_{s=1}^{m}|x_{ms}| \to a<\infty$, $(1/(m\log m))\sum_{s\leqslant m}|x_{ms}||\log|x_s|| \to 0$, *and* $\sup_{s\leqslant m}(|x_{ms}|/m)\to 0$.
Then $(1/(m\log m))\sum_{s=1}^{m}(|x_{ms}|/|\zeta_s|)\to_p 2f(0)a$ as $m\to\infty$.

Proof of Lemma C. Define $\xi_{ms} = |x_{ms}|/(|\zeta_s| m\log m)$, $\xi'_{ms} = \xi_{ms}\mathbf{1}(\xi_{ms}<1)$ and $\xi''_{ms} = \xi_{ms}\mathbf{1}(\xi_{ms}\geqslant 1)$, where $\mathbf{1}(A)$ is the indicator of A. Let $b_{ms} = E\xi'_{ms}$ and $f^* = \sup_x f(x)$. The assertion follows if, as $m\to\infty$: (i) $\sum_{s=1}^{m} b_{ms}\to 2f(0)a$; (ii) $\sum_{s=1}^{m}\xi''_{ms}\to_p 0$; and (iii) $\sum_{s=1}^{m}(\xi'_{ms}-b_{ms})\to_p 0$.

First, the symmetry condition for f in (A1) implies that the probability density of ξ_{ms} is

$$2f\left(\frac{|x_{ms}|}{xm\log m}\right)\frac{|x_{ms}|}{x^2 m\log m}.$$

Hence, by integration by parts and the first part of (A2)

$$\begin{aligned}
b_{ms} &= 2\int_0^1 xf\left(\frac{|x_{ms}|}{xm\log m}\right)\frac{|x_{ms}|}{x^2 m\log m}\,\mathrm{d}x\\
&= 2\frac{|x_{ms}|}{m\log m}\int_0^{m\log m/|x_{ms}|} f\left(\frac{1}{y}\right)\frac{\mathrm{d}y}{y}\\
&= 2\frac{|x_{ms}|}{m\log m}\left[\log\left(\frac{m\log m}{|x_{ms}|}\right)f\left(\frac{|x_{ms}|}{m\log m}\right) + \int_0^{m\log m/|x_{ms}|}\log(y)f'\left(\frac{1}{y}\right)\frac{\mathrm{d}y}{y^2}\right].
\end{aligned}$$

Since, by the second part of (A2) and the continuity of f',

$$\left|\int_0^{m\log m/|x_{ms}|}\log(y)f'\left(\frac{1}{y}\right)\frac{\mathrm{d}y}{y^2}\right| \leqslant \int_0^{\infty}\left|f'\left(\frac{1}{y}\right)\frac{\log(y)}{y^2}\right|\mathrm{d}y = \int_0^{\infty}|f'(x)\log x|\,\mathrm{d}x<\infty,$$

it follows from (A1)–(A3) that

$$\sum_{s=1}^{m} b_{ms} = \frac{2}{m\log m}\sum_{s=1}^{m} |x_{ms}| f\left(\frac{|x_{ms}|}{m\log m}\right)\{\log m + \log\log m - \log|x_{ms}|\} + \mathrm{O}(1)$$
$$\to 2f(0)a \text{ as } \quad m \to \infty, \tag{16}$$

which yields (i). For part (ii), given some small $\varepsilon > 0$

$$\begin{aligned} P\left(\sum_{s=1}^{m} \xi''_{ms} > \varepsilon\right) &\leqslant \sum_{s=1}^{m} P\left(\frac{|x_{ms}|}{|\zeta_s|}\mathbf{1}\left(\frac{|x_{ms}|}{|\zeta_s|} \geqslant m\log m\right) > \varepsilon m\log m\right) \\ &= \sum_{s=1}^{m} P\left(\frac{|x_{ms}|}{|\zeta_s|} \geqslant m\log m\right) \\ &= \sum_{s=1}^{m} P\left(|\zeta_s| \leqslant \frac{|x_{ms}|}{m\log m}\right) \\ &\leqslant 2f^* \sum_{s=1}^{m} \frac{|x_{ms}|}{m\log m}, \end{aligned}$$

which by (A3) converges to 0 as $m \to \infty$. Part (ii) follows. Next, by Chebyshev's inequality,

$$P\left(\sum_{s=1}^{m}(\xi'_{ms} - b_{ms}) > \varepsilon\right) \leqslant \varepsilon^{-2}\sum_{s=1}^{m}(E\xi'^2_{ms} - b^2_{ms}).$$

Clearly, by (A3) and (16)

$$\sum_{s=1}^{m} b^2_{ms} \leqslant \max_{s\leqslant m} b_{ms}\left(\sum_{s=1}^{m} b_{ms}\right) \to 0 \quad \text{as } m \to \infty.$$

In addition, (A3) gives

$$\begin{aligned} \sum_{s=1}^{m} E\xi'^2_{ms} &= 2\sum_{s=1}^{m}\int_0^1 y^2 f\left(\frac{|x_{ms}|}{ym\log m}\right)\frac{|x_{ms}|}{y^2 m\log m}\,\mathrm{d}y \\ &= 2\sum_{s=1}^{m}\frac{|x_{ms}|^2}{(m\log m)^2}\int_{m\log m/|x_{ms}|}^{\infty} f(x)x^{-2}\,\mathrm{d}x \\ &\leqslant 2f^*\frac{1}{m\log m}\sum_{s=1}^{m}|x_{ms}| \to 0 \quad \text{as } m \to \infty, \end{aligned}$$

giving part (iii). The proof is then complete. □

Proof of Lemma 3.1. We prove the strong approximation first. In view of Assumption L, we may use the BN decomposition (see Phillips and Solo, 1992) to write

$$C(L) = C(1) + \widetilde{C}(L)(L-1),$$

where $\widetilde{C}(L) = \sum_{j=0}^{\infty}\widetilde{c}_j L^j$ with $\widetilde{c}_j = \sum_{s=j+1}^{\infty} c_s$ and $\sum_{j=0}^{\infty}|\widetilde{c}_j| < \infty$. Then,

$$u_t = C(1)\varepsilon_t + \widetilde{\varepsilon}_{t-1} - \widetilde{\varepsilon}_t,$$

with $\widetilde{\varepsilon}_t = \widetilde{C}(L)\varepsilon_t$, and

$$S_t = C(1)\sum_{j=1}^{t} \varepsilon_j + \widetilde{\varepsilon}_0 - \widetilde{\varepsilon}_t = S_{\eta t} + \widetilde{\varepsilon}_0 - \widetilde{\varepsilon}_t,$$

where $S_{\eta t} = \sum_{j=1}^{t} \eta_j$ and $\eta_j = C(1)\varepsilon_j$. Next, since η_j is iid with mean zero and finite moments of order $q > 2p > 4$, we may use a strong approximation to the partial sum process $S_{\eta t}$ of η_j. In particular, by a result of Komlós et al. (1976) we can expand the probability space as necessary to set up a partial sum process that is distributionally equivalent to $S_{\eta k}$ and a Brownian motion $B(\cdot)$ with variance $2\pi f_u(0)$ on the same space for which

$$\sup_{0 \leqslant k \leqslant n} |S_{\eta k} - B(k)| = o_{a.s}(n^{1/q}), \tag{17}$$

giving a uniform approximation to $S_{\eta k}$ over $0 \leqslant k \leqslant n$ in terms of the Brownian motion B. Next, since

$$|S_k - B(k)| \leqslant |S_{\eta k} - B(k)| + |\widetilde{\varepsilon}_0 - \widetilde{\varepsilon}_k|,$$

we have

$$\sup_{0 \leqslant k \leqslant n} \left| \frac{S_k}{\sqrt{n}} - B\left(\frac{k}{n}\right) \right| \leqslant \sup_{0 \leqslant k \leqslant n} \left| \frac{S_{\eta k}}{\sqrt{n}} - B\left(\frac{k}{n}\right) \right| + 2 \sup_{0 \leqslant k \leqslant n} \frac{|\widetilde{\varepsilon}_k|}{\sqrt{n}}. \tag{18}$$

Now

$$\sup_{0 \leqslant k \leqslant n} \frac{|\widetilde{\varepsilon}_k|}{\sqrt{n}} = o_{a.s.}\left(\frac{1}{n^{1/2-1/p}}\right) \tag{19}$$

holds if

$$\max_{0 \leqslant k \leqslant n} \frac{|\widetilde{\varepsilon}_k|}{n^{1/p}} = o_{a.s.}(1). \tag{20}$$

But

$$\begin{aligned}
\mathrm{P}\left[\max_{0 \leqslant k \leqslant n} \frac{|\widetilde{\varepsilon}_k|}{n^{1/p}} > \delta\right] &= \mathrm{P}\left[\max_{0 \leqslant k \leqslant n} \frac{|\widetilde{\varepsilon}_k|^q}{n^{q/p}} > \delta^q\right] \\
&= \mathrm{P}\left[\frac{1}{n^{q/p}} \sum_{k=1}^{n} |\widetilde{\varepsilon}_k|^q \mathbf{1}[|\widetilde{\varepsilon}_k|^q > n^{q/p}\delta^q] > \delta^q\right] \\
&< \frac{\mathrm{E}\left(\sum_{k=1}^{n} |\widetilde{\varepsilon}_k|^q \mathbf{1}[|\widetilde{\varepsilon}_k|^q > n^{q/p}\delta^q]\right)}{n^{q/p}\delta^q} \\
&= \frac{\mathrm{E}(|\widetilde{\varepsilon}_k|^q \mathbf{1}[|\widetilde{\varepsilon}_k|^q > n^{q/p}\delta^q])}{n^{q/p-1}\delta^q},
\end{aligned}$$

by stationarity of $\widetilde{\varepsilon}_k$. It follows that

$$\sum_{n=1}^{\infty} \mathrm{P}\left[\max_{0 \leqslant k \leqslant n} \frac{|\widetilde{\varepsilon}_k|}{n^{1/p}} > \delta\right] < \sum_{n=1}^{\infty} \frac{\mathrm{E}(|\widetilde{\varepsilon}_k|^q)}{n^{q/p-1}} < \infty$$

since $q > 2p$. Result (20) then follows by the Borel Cantelli lemma. We deduce from (17)–(19) that

$$\sup_{0 \leqslant k \leqslant n} \left| \frac{S_k}{\sqrt{n}} - B\left(\frac{k}{n}\right) \right| = o_{a.s.}\left(\frac{1}{n^{1/2-1/p}}\right),$$

giving the stated result.

Now suppose only pth moments are finite and we need to prove the uniform approximation (7) holds in probability. Then, (17) still holds for $q = p > 2$ by a result of Major (1976) and, in place of (20), the following is sufficient:

$$\max_{0 \leqslant k \leqslant n} \frac{|\widetilde{\varepsilon}_k|}{n^{1/p}} = \mathrm{o_p}(1). \tag{21}$$

Apparently, (21) holds if

$$\begin{aligned}
\mathrm{P}\left[\max_{0 \leqslant k \leqslant n} \frac{|\widetilde{\varepsilon}_k|}{n^{1/p}} > \delta\right] &= \mathrm{P}\left[\max_{0 \leqslant k \leqslant n} \frac{|\widetilde{\varepsilon}_k|^p}{n} > \delta^p\right] \\
&= \mathrm{P}\left[\frac{1}{n}\sum_{k=1}^{n} |\widetilde{\varepsilon}_k|^p \mathbf{1}[|\widetilde{\varepsilon}_k|^p > n\delta^p] > \delta^p\right] \\
&< \frac{\mathrm{E}(\sum_{k=1}^{n} |\widetilde{\varepsilon}_k|^p \mathbf{1}[|\widetilde{\varepsilon}_k|^p > n\delta^p])}{n\delta^p} \\
&= \frac{\mathrm{E}(|\widetilde{\varepsilon}_k|^p \mathbf{1}[|\widetilde{\varepsilon}_k|^p > n\delta^p])}{\delta^p} \to 0,
\end{aligned}$$

which will be so when $\mathrm{E}(|\widetilde{\varepsilon}_k|^p) < \infty$. By Minkowski's inequality, we have

$$\begin{aligned}
\mathrm{E}(|\widetilde{\varepsilon}_k|^p) &= \mathrm{E}\left(\left|\sum_{j=0}^{\infty} \widetilde{c}_j \varepsilon_{k-j}\right|^p\right) < \left(\sum_{j=0}^{\infty} (|\widetilde{c}_j|^p \mathrm{E}|\varepsilon_{k-j}|^p)^{1/p}\right)^p \\
&= \left(\sum_{j=0}^{\infty} |\widetilde{c}_j|\right)^p \mathrm{E}|\varepsilon_k|^p < \left(\sum_{j=0}^{\infty} j|c_j|\right)^p \mathrm{E}|\varepsilon_k|^p.
\end{aligned}$$

Thus, (7) holds under the weaker moment condition $\mathrm{E}(|\varepsilon_k|^p) < \infty$ for any $p > 2$. □

Proof of Theorem 3.2. By virtue of (5) or by using partial summation for a direct calculation and using the differencing operator $\Delta = 1 - L$, we have for $s = 1, \ldots, m$

$$\begin{aligned}
w_u(\lambda_s) &= \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} \mathrm{e}^{2\pi \mathrm{i} st/n} u_t = \Delta\left(\frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} \mathrm{e}^{2\pi \mathrm{i} st/n} S_t\right) - \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} S_{t-1} \Delta(\mathrm{e}^{2\pi \mathrm{i} st/n}) \\
&= \frac{1}{\sqrt{2\pi}} \frac{S_n}{\sqrt{n}} - \frac{1}{\sqrt{2\pi}} \sum_{t=1}^{n} \frac{S_{t-1}}{\sqrt{n}} \mathrm{e}^{2\pi \mathrm{i} s(t-1)/n} (\mathrm{e}^{2\pi \mathrm{i} s/n} - 1).
\end{aligned} \tag{22}$$

In view of the embedding (7), we have

$$\frac{S_{t-1}}{\sqrt{n}} = B\left(\frac{t-1}{n}\right) + \mathrm{o_p}\left(\frac{1}{n^{1/2-1/p}}\right),$$

where the error magnitude holds uniformly in $t = 1, \ldots, n$. Then, we have

$$\begin{aligned}
w_u(\lambda_s) &= \frac{1}{\sqrt{2\pi}} \left[B(1) + \mathrm{o_p}\left(\frac{1}{n^{1/2-1/p}}\right)\right] \\
&\quad - \frac{1}{\sqrt{2\pi}} \sum_{t=1}^{n} \left[B\left(\frac{t-1}{n}\right) + \mathrm{o_p}\left(\frac{1}{n^{1/2-1/p}}\right)\right] \mathrm{e}^{2\pi \mathrm{i} s(t-1)/n} (\mathrm{e}^{2\pi \mathrm{i} s/n} - 1)
\end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}} B(1) - \frac{1}{\sqrt{2\pi}} \sum_{t=1}^{n} \left[B\left(\frac{t-1}{n}\right) + \mathrm{o_p}\left(\frac{1}{n^{1/2-1/p}}\right) \right]$$
$$\times \mathrm{e}^{2\pi \mathrm{i} s(t-1)/n} (\mathrm{e}^{2\pi \mathrm{i} s/n} - 1) + \mathrm{o_p}\left(\frac{1}{n^{1/2-1/p}}\right), \tag{23}$$

where the error magnitude holds uniformly in $s \leqslant m$.

Next write (23) in the form

$$w_u(\lambda_s) = \frac{1}{\sqrt{2\pi}} B(1) - \frac{1}{\sqrt{2\pi}} \sum_{t=1}^{n} B\left(\frac{t-1}{n}\right) \mathrm{e}^{2\pi \mathrm{i} s(t-1)/n} (\mathrm{e}^{2\pi \mathrm{i} s/n} - 1) + \mathrm{o_p}\left(\frac{s}{n^{1/2-1/p}}\right)$$

and since

$$\mathrm{e}^{2\pi \mathrm{i} s(t-1)/n} (\mathrm{e}^{2\pi \mathrm{i} s/n} - 1) = 2\pi \mathrm{i} s \int_{(t-1)/n}^{t/n} \mathrm{e}^{2\pi \mathrm{i} s r} \, \mathrm{d}r,$$

we have

$$\sum_{t=1}^{n} B\left(\frac{t-1}{n}\right) \mathrm{e}^{2\pi \mathrm{i} s(t-1)/n} (\mathrm{e}^{2\pi \mathrm{i} s/n} - 1)$$
$$= 2\pi \mathrm{i} s \sum_{t=1}^{n} B\left(\frac{t-1}{n}\right) \int_{(t-1)/n}^{t/n} \mathrm{e}^{2\pi \mathrm{i} s r} \, \mathrm{d}r$$
$$= 2\pi \mathrm{i} s \sum_{t=1}^{n} \int_{(t-1)/n}^{t/n} B\left(\frac{[nr]}{n}\right) \mathrm{e}^{2\pi \mathrm{i} s r} \, \mathrm{d}r = 2\pi \mathrm{i} s \int_0^1 B\left(\frac{[nr]}{n}\right) \mathrm{e}^{2\pi \mathrm{i} s r} \, \mathrm{d}r.$$

Next, set

$$v_{s,n} = \int_0^1 \left\{ B\left(\frac{[nr]}{n}\right) - B(r) \right\} \mathrm{e}^{2\pi \mathrm{i} s r} \, \mathrm{d}r.$$

Then

$$w_u(\lambda_s) = \frac{1}{\sqrt{2\pi}} B(1) - \frac{2\pi \mathrm{i} s}{\sqrt{2\pi}} \int_0^1 B(r) \mathrm{e}^{2\pi \mathrm{i} s r} \, \mathrm{d}r - \frac{2\pi \mathrm{i} s}{\sqrt{2\pi}} v_{s,n} + \mathrm{o_p}\left(\frac{s}{n^{1/2-1/p}}\right)$$
$$= \frac{1}{\sqrt{2\pi}} \int_0^1 \mathrm{e}^{2\pi \mathrm{i} s r} \, \mathrm{d}B(r) - \frac{2\pi \mathrm{i} s}{\sqrt{2\pi}} v_{s,n} + \mathrm{o_p}\left(\frac{s}{n^{1/2-1/p}}\right)$$
$$= \frac{1}{\sqrt{2\pi}} \int_0^1 \mathrm{e}^{2\pi \mathrm{i} s r} \mathrm{d}B(r) + \mathrm{o_p}\left(\frac{m}{n^{1/2-1/p}}\right),$$

since $v_{s,n} = \mathrm{O_p}(n^{-1/2})$ uniformly in $s = 0, \ldots, n$. Indeed,

$$|v_{s,n}| \leqslant \int_0^1 \left| B\left(\frac{[nr]}{n}\right) - B(r) \right| \mathrm{d}r,$$

and hence

$$\mathrm{E}|v_{s,n}| \leqslant \int_0^1 \mathrm{E}\left|B\left(\frac{[nr]}{n}\right) - B(r)\right| \mathrm{d}r \leqslant \int_0^1 \left(\mathrm{E}\left\{B\left(\frac{[nr]}{n}\right) - B(r)\right\}^2\right)^{1/2} \mathrm{d}r$$
$$= \int_0^1 \left|\frac{[nr]}{n} - r\right|^{1/2} \mathrm{d}r \leqslant Cn^{-1/2},$$

for come constant C, which implies that $v_{s,n} = \mathrm{O_p}(n^{-1/2})$ uniformly in s.

When $s = 0$, we have

$$\frac{X_n}{\sqrt{2\pi n}} = \frac{1}{\sqrt{2\pi}} B(1) + \mathrm{o_p}\left(\frac{1}{n^{1/2-1/p}}\right).$$

Now use the notation

$$\xi_s = \frac{1}{\sqrt{2\pi}} \int_0^1 \mathrm{e}^{2\pi \mathrm{i} sr} \, \mathrm{d}B(r), \quad \eta = \frac{1}{\sqrt{2\pi}} B(1). \tag{24}$$

The variates $\{\xi_s\}_{s=1}^m$ are independent complex Gaussian $\mathrm{N_c}(0, f_u(0))$ and are independent of η, which is real Gaussian $\mathrm{N}(0, f_u(0))$, thereby delivering the stated result. $\square$

Proof of Theorem 3.3. From (6)

$$\log |w_X(\lambda_s)|^2 = -2 \log |1 - \mathrm{e}^{\mathrm{i}\lambda_s}| + \log \left| w_u(\lambda_s) - \mathrm{e}^{\mathrm{i}\lambda_s} \frac{X_n}{\sqrt{2\pi n}} \right|^2,$$

and using (3) we have

$$2\sqrt{m}(\widehat{d} - 1) = -\frac{(1/\sqrt{m})\sum_{s=1}^m x_s \log |w_u(\lambda_s) - (1/\sqrt{2\pi n})\mathrm{e}^{\mathrm{i}\lambda_s} X_n|^2}{(1/m)\sum_{s=1}^m x_s^2}. \tag{25}$$

From Theorem 3.2 we have

$$w_u(\lambda_s) = \frac{1}{\sqrt{2\pi}} \int_0^1 \mathrm{e}^{2\pi \mathrm{i} sr} \, \mathrm{d}B(r) + \mathrm{o_p}\left(\frac{m}{n^{1/2-1/p}}\right), \tag{26}$$

which holds uniformly in $s \leqslant m$. Using the notation of (24) above, the variates $\{\xi_s\}_{s=1}^m$ are independent complex Gaussian $\mathrm{N_c}(0, f_u(0))$ and are independent of η, which is real Gaussian $\mathrm{N}(0, f_u(0))$. Write

$$w_u(\lambda_s) - \frac{1}{\sqrt{2\pi n}} \mathrm{e}^{\mathrm{i}\lambda_s} X_n = \xi_s - \eta + A_s, \tag{27}$$

where $A_s = \mathrm{o_p}(m/n^{1/2-1/p})$ uniformly in $s \leqslant m$. Then, we have

$$\frac{1}{\sqrt{m}} \sum_{s=1}^m x_s \log \left| w_u(\lambda_s) - \frac{1}{\sqrt{2\pi n}} \mathrm{e}^{\mathrm{i}\lambda_s} X_n \right|^2$$
$$= \frac{1}{\sqrt{m}} \sum_{s=1}^m x_s \log |\xi_s - \eta + A_s|^2$$
$$= \frac{1}{\sqrt{m}} \sum_{s=1}^m x_s \left[\log |\xi_s - \eta|^2 + \log \left| 1 + \frac{A_s}{\xi_s - \eta} \right|^2 \right]. \tag{28}$$

Using the inequality

$$|\log|1+a||\leqslant|a|+\frac{|a|}{|1+a|},$$

we get

$$\begin{aligned}\left|\frac{1}{\sqrt{m}}\sum_{s=1}^{m}x_s\log\left|1+\frac{A_s}{\xi_s-\eta}\right|^2\right|&\leqslant\frac{2}{\sqrt{m}}\sum_{s=1}^{m}|x_s|\left\{\frac{|A_s|}{|\xi_s-\eta|}\left[1+\frac{1}{|1+A_s/(\xi_s-\eta)|}\right]\right\}\\&=\frac{2}{\sqrt{m}}\sum_{s=1}^{m}|x_s|\frac{|A_s|}{|\xi_s-\eta|}B_s,\end{aligned}\tag{29}$$

where

$$B_s=1+\frac{1}{|1+A_s/(\xi_s-\eta)|}.$$

Now $B_s=\mathrm{O_p}(1)$ uniformly in $s\leqslant m$ if

$$\max_{s\leqslant m}\left|\frac{A_s}{\xi_s-\eta}\right|\to_p 0\tag{30}$$

because then $1+A_s/(\xi_s-\eta)\to 1$ uniformly in $s\leqslant m$. Note that

$$\begin{aligned}\max_{s\leqslant m}\left|\frac{A_s}{\xi_s-\eta}\right|&\leqslant\max_{s\leqslant m}|A_s|\max_{s\leqslant m}\left|\frac{1}{\xi_s-\eta}\right|\\&\leqslant(m\log m)\max_{s\leqslant m}|A_s|\left[\frac{1}{m\log m}\max_{s\leqslant m}\left|\frac{1}{\xi_s-\eta}\right|\right].\end{aligned}$$

Now

$$\frac{1}{|\xi_s-\eta|}=\frac{1}{[(\xi_{sa}-\eta)^2+\xi_{sb}^2]^{1/2}}\leqslant\frac{1}{|\xi_{sb}|}=\frac{2}{|\zeta_s|},\tag{31}$$

where ζ_s is iid $\mathrm{N}(0,1)$, and so

$$\begin{aligned}\max_{s\leqslant m}\left|\frac{A_s}{\xi_s-\eta}\right|&\leqslant\max_{s\leqslant m}|(m\log m)A_s|\left[\frac{2}{m\log m}\sum_{s\leqslant m}\frac{1}{|\zeta_s|}\right]\\&=\mathrm{O_p}\left((m\log m)\max_{s\leqslant m}|A_s|\right),\end{aligned}$$

by application of Lemma C with $x_s=1$ for all s. Further,

$$(m\log m)A_s=\mathrm{o_p}\left(\frac{m^2\log m}{n^{1/2-1/p}}\right)\tag{32}$$

uniformly in $s\leqslant m$ as $n\to\infty$ so that

$$\max_{s\leqslant m}(m\log m)|A_s|=\mathrm{o_p}(1)\tag{33}$$

as $n\to\infty$ under (9). Hence,

$$\max_{s\leqslant m}B_s=\mathrm{O_p}(1).\tag{34}$$

Using (31) and (34) we have

$$\frac{2}{\sqrt{m}}\sum_{s=1}^{m}|x_s|\frac{|A_s|}{|\xi_s-\eta|}B_s\leqslant\frac{4}{\sqrt{m}}\sum_{s=1}^{m}|x_s|\frac{|A_s|}{|\zeta_s|}\times \mathrm{O_p}(1). \tag{35}$$

Since $A_s=\mathrm{o_p}(m/n^{1/2-1/p})$ uniformly in $s\leqslant m$, for any $\varepsilon>0$, however small, there exists $C_\varepsilon>0$ such that with probability exceeding $1-\varepsilon$

$$|A_s|\leqslant C_\varepsilon\frac{m}{n^{1/2-1/p}},$$

uniformly in $s\leqslant m$. Then, with probability exceeding $1-\varepsilon$

$$\frac{4}{\sqrt{m}}\sum_{s=1}^{m}|x_s|\frac{|A_s|}{|\zeta_s|}\leqslant\frac{4\sqrt{m}}{n^{1/2-1/p}}\sum_{s=1}^{m}|x_s|\frac{C_\varepsilon}{|\zeta_s|}.$$

Since $x_s=a_s-\overline{a}$, where $\overline{a}=m^{-1}\sum_{s=1}^{m}a_s$ and $a_s=\log|1-\mathrm{e}^{\mathrm{i}\lambda_s}|=\log(2\sin(\lambda_s/2))=\log\lambda_s-(\frac{1}{24})\lambda_s^2+\mathrm{O}(\lambda_s^4)$, we find that

$$x_s=\log\frac{s}{m}-\frac{1}{m}\sum_{s=1}^{m}\log\frac{s}{m}+\mathrm{O}\left(\frac{m^2}{n^2}\right), \tag{36}$$

uniformly for $s\leqslant m$. Then, using Euler summation it follows that

$$\begin{aligned}x_s&=\log\frac{s}{m}-\int_{1/m}^{1}\log r\,\mathrm{d}r+\mathrm{O}\left(\frac{\log m}{m}\right)\\&=1+\log\frac{s}{m}+\mathrm{O}\left(\frac{\log m}{m}\right),\end{aligned} \tag{37}$$

from which we deduce that

$$\frac{1}{m}\sum_{s=1}^{m}|x_s|\to\int_0^1|1+\log r|\,\mathrm{d}r<\infty,$$

$$\max_{s\leqslant m}\frac{|x_s|}{m}\to 0,$$

and

$$\begin{aligned}&\frac{1}{m\log m}\sum_{s=1}^{m}|x_s|\log|x_s|\\&\quad=\mathrm{O}\left(\frac{1}{\log m}\int_{1/m}^{1}\left|1+\log r+\mathrm{O}\left(\frac{\log m}{m}\right)\right|\log\left|1+\log r+\mathrm{O}\left(\frac{\log m}{m}\right)\right|\mathrm{d}r\right)\\&\quad=\mathrm{O}\left(\frac{1}{\log m}\int_{1}^{m}\frac{1}{x^2}\log x\,\log\,\log\,x\,\mathrm{d}x\right)=\mathrm{O}\left(\frac{1}{\log m}\right)\to 0,\end{aligned}$$

as $m\to\infty$. Hence, from Lemma C, we find

$$\frac{1}{m\log m}\sum_{s=1}^{m}\frac{|x_s|}{|\zeta_s|}=\mathrm{O_p}(1),$$

and it follows that

$$\frac{4}{\sqrt{m}}\sum_{s=1}^{m}|x_s|\frac{|A_s|}{|\zeta_s|}=\mathrm{O_p}\left(\frac{m^{3/2}\log m}{n^{1/2-1/p}}\right). \tag{38}$$

From (29), (35) and (38) we deduce that

$$\left|\frac{1}{\sqrt{m}}\sum_{s=1}^{m}x_s\log\left|1+\frac{A_s}{\xi_s-\eta}\right|^2\right|=\mathrm{O_p}\left(\frac{m^{3/2}\log m}{n^{1/2-1/p}}\right), \tag{39}$$

and so (28) is

$$\frac{1}{\sqrt{m}}\sum_{s=1}^{m}x_s\log|\xi_s-\eta|^2+\mathrm{O_p}\left(\frac{m^{3/2}\log m}{n^{1/2-1/p}}\right). \tag{40}$$

From Lemma B, we have

$$\frac{1}{\sqrt{m}}\sum_{j=1}^{m}x_s\log|\xi_s-\eta|^2\rightarrow_d\int_0^\infty \mathrm{N}(0,\sigma^2(w))\mathrm{pdf}(w)\,\mathrm{d}w, \tag{41}$$

and so, when (9) holds, we deduce that

$$\frac{1}{\sqrt{m}}\sum_{s=1}^{m}x_s\log\left|w_u(\lambda_s)-\frac{1}{\sqrt{2\pi n}}\mathrm{e}^{\mathrm{i}\lambda_s}X_n\right|^2\rightarrow_d\int_0^\infty \mathrm{N}(0,\sigma^2(w))\mathrm{pdf}(w)\,\mathrm{d}w. \tag{42}$$

Finally, using (42) and $m^{-1}\sum_{s=1}^{m}x_s^2\rightarrow 1$ in (25), we obtain

$$\sqrt{m}(\widehat{d}-1)\rightarrow_d MN\left(0,\frac{1}{4}\sigma^2(W)\right)\equiv\int_0^\infty \mathrm{N}\left(0,\frac{1}{4}\sigma^2(w)\right)\mathrm{pdf}(w)\,\mathrm{d}w,$$

giving the required result.

When u_t is Gaussian, we use the frequency domain BN decomposition (Phillips and Solo, 1992, p. 986)

$$w_u(\lambda_s)=C(\mathrm{e}^{\mathrm{i}\lambda_s})w_\varepsilon(\lambda_s)+\frac{1}{\sqrt{2\pi n}}(\widetilde{\varepsilon}_{0\lambda_s}-\widetilde{\varepsilon}_{n\lambda_s}),$$

where $\widetilde{\varepsilon}_{t\lambda_s}=\sum_{j=0}^{\infty}\widetilde{c}_{j\lambda_s}\varepsilon_{t-j}$ and $\widetilde{c}_{j\lambda_s}=\mathrm{e}^{-\mathrm{i}\lambda_s j}\sum_{k=j+1}^{\infty}c_k\mathrm{e}^{\mathrm{i}\lambda_s k}$. Under (4), $\sum_{j=0}^{\infty}|\widetilde{c}_{j\lambda_s}|<\infty$, $\widetilde{\varepsilon}_{t\lambda_s}=\mathrm{O_p}(1)$ and so

$$w_u(\lambda_s)=C(1)w_\varepsilon(\lambda_s)+\mathrm{O_p}\left(\frac{1}{\sqrt{n}}+\frac{m}{n}\right)=C(1)w_\varepsilon(\lambda_s)+\mathrm{O_p}\left(\frac{1}{\sqrt{n}}\right),$$

uniformly over $s\leqslant m$ when

$$\frac{m\log m}{n^{1/2}}\rightarrow 0. \tag{43}$$

It follows that (27) now holds with $A_s=\mathrm{O_p}(1/n^{1/2})$ uniformly in $s\leqslant m$. The remainder of the earlier proof applies with this order of magnitude in place of $\mathrm{o_p}(m/n^{1/2-1/p})$. In particular, we have

$$\max_{s\leqslant m}(m\log m)|A_s|=\mathrm{O_p}\left(\frac{m\log m}{n^{1/2}}\right)$$

in place of (32). Then, (30) and (34) follow under (43). We deduce that

$$\frac{4}{\sqrt{m}}\sum_{s=1}^{m}|x_s|\frac{|A_s|}{|\zeta_s|}=\mathrm{O_p}\left(\frac{m^{1/2}\log m}{n^{1/2}}\right),$$

and

$$\left|\frac{1}{\sqrt{m}}\sum_{s=1}^{m}x_s\log\left|1+\frac{A_s}{\xi_s-\eta}\right|^2\right|=\mathrm{O_p}\left(\frac{m^{1/2}\log m}{n^{1/2}}\right)$$

in place of (38) and (39). Result (42) therefore follows under (43), giving the required result. $\square$

Proof of Corollary 3.4. The variance of the limit distribution is $\sigma_d^2=(1/4)\int_0^\infty\sigma^2(w)$ $\mathrm{pdf}(w)\,\mathrm{d}w$, and, using expression (11) for $\sigma^2(w)$, we calculate

$$\begin{aligned}
&\int_0^\infty \sigma^2(w)\mathrm{pdf}(w)\,\mathrm{d}w\\
&=\frac{1}{2^{1/2}\Gamma(1/2)}\int_0^\infty\left(\mathrm{e}^{-w}\sum_{j=0}^{\infty}\frac{w^j}{j!}\{\psi(1+j)^2+\psi'(1+j)\}-\left[\mathrm{e}^{-w}\sum_{j=0}^{\infty}\frac{w^j}{j!}\psi(1+j)\right]^2\right)\\
&\quad\times\mathrm{e}^{-w/2}w^{-1/2}\,\mathrm{d}w\\
&=\frac{1}{\sqrt{2\pi}}\sum_{j=0}^{\infty}\frac{\{\psi(1+j)^2+\psi'(1+j)\}}{j!}\int_0^\infty\mathrm{e}^{-(3/2)w}w^{j-1/2}\,\mathrm{d}w\\
&\quad-\frac{1}{\sqrt{2\pi}}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\psi(1+j)\psi(1+k)}{j!k!}\int_0^\infty\mathrm{e}^{-(5/2)w}w^{j+k-1/2}\,\mathrm{d}w\\
&=\frac{1}{\sqrt{2\pi}}\sum_{j=0}^{\infty}\frac{\{\psi(1+j)^2+\psi'(1+j)\}}{j!}\frac{\Gamma(j+1/2)}{(3/2)^{j+1/2}}\\
&\quad-\frac{1}{\sqrt{2\pi}}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\psi(1+j)\psi(1+k)}{j!k!}\frac{\Gamma(j+k+1/2)}{(5/2)^{j+k+1/2}}\\
&=\frac{\pi^2}{6}+\frac{1}{\sqrt{3}}\sum_{j=0}^{\infty}\frac{[\psi(1+j)^2-\sum_{k=1}^{j}(1/k^2)](1/2)_j(2/3)^j}{j!}\\
&\quad-\frac{1}{\sqrt{5}}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\psi(1+j)\psi(1+k)}{j!k!}\left(\frac{1}{2}\right)_{j+k}\left(\frac{2}{5}\right)^{j+k},
\end{aligned}\tag{44}$$

since

$$\frac{1}{\sqrt{2\pi}}\sum_{j=0}^{\infty}\frac{\psi(1+j)^2}{j!}\frac{\Gamma(j+1/2)}{(3/2)^{j+1/2}}=\frac{1}{\sqrt{3}}\sum_{j=0}^{\infty}\frac{\psi(1+j)^2(1/2)_j(2/3)^j}{j!},$$

$$\psi'(1+j)=\frac{\pi^2}{6}-\sum_{k=1}^{j}\frac{1}{k^2},\tag{45}$$

and

$$\psi(1+j)=\begin{cases}-C+\sum_{k=1}^{j}\frac{1}{k}, & j\geqslant 1\\ -C, & j=0\end{cases}\quad C=0.577215 \text{ (Euler' s constant)}, \tag{46}$$

where (45) and (46) are standard—e.g. Gradshteyn and Ryzhik (1994, 8.365 and 8.366). The summation in the third term of (44) can be written as

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\psi(1+j)\psi(1+k)}{j!k!}\left(\frac{1}{2}\right)_{j+k}\left(\frac{2}{5}\right)^{j+k}$$
$$=\sum_{j=0}^{\infty}\frac{\psi(1+j)(1/2)_j(2/5)^j}{j!}\sum_{k=0}^{\infty}\frac{\psi(1+k)}{k!}\left(\frac{1}{2}+j\right)_k\left(\frac{2}{5}\right)^k.$$

Hence, the variance of the limit distribution of $\sqrt{m}(\widehat{d}-d)$ is

$$\sigma_d^2=\frac{\pi^2}{24}+\frac{1}{4\sqrt{3}}\sum_{j=0}^{\infty}\frac{[\psi(1+j)^2-\sum_{k=1}^{j}(1/k^2)](1/2)_j(2/3)^j}{j!}$$
$$-\frac{1}{4\sqrt{5}}\sum_{j=0}^{\infty}\frac{\psi(1+j)(1/2)_j(2/5)^j}{j!}\sum_{k=0}^{\infty}\frac{\psi(1+k)}{k!}\left(\frac{1}{2}+j\right)_k\left(\frac{2}{5}\right)^k,$$

giving the stated result. □

References

Akonom, J., 1993. Comportement asymptotique du temps d'occupation du processus des sommes partielles. Annals of the Institute of Henri Poincaré 29, 57–81.

Baillie, R.T., 1996. Long memory processes and fractional integration in econometrics. Journal of Econometrics 73, 5–59.

Chan, N.H., Wei, C.Z., 1987. Asymptotic inference for nearly nonstationary AR(1) processes. Annals of Statistics 15, 1050–1063.

Fay, G., Soulier, P., 1999. The periodogram of an iid sequence. Université d'Evry Val d'Essonne. Prepublicatoin 112.

Geweke, J., Porter-Hudak, S., 1983. The estimation and application of long memory time series models. Journal of Time Series Analysis 4, 221–237.

Giraitis, L., Phillips, P.C.B., 2006. Uniform limit theory for stationary autoregression. Journal of Time Series Analysis 27, 51–60.

Gradshteyn, I.S., Ryzhik, I.M., 1994. Tables of Integrals, Series and Products, fifth ed. Academic Press, New York.

Hannan, E.J., 1973. Central limit theorems for time series regression. Zeitschrift für Wahrscheinlichkeitsheorie und verwandte Gebiete 26, 157–170.

Hurvich, C.M., Beltrao, K.I., 1993. Automatic semiparametric estimation of the memory parameter of a long-memory time series. Journal of Time Series Analysis 15, 285–302.

Hurvich, C.M., Ray, B.K., 1995. Estimation of the memory parameter for nonstationary or noninvertible fractionally integrated processes. Journal of Time Series Analysis 16, 17–42.

Hurvich, C.M., Deo, R., Brodsky, J., 1998. The mean squared error of Geweke and Porter Hudak's estimator of the memory parameter of a long memory time series. Journal of Time Series Analysis 19, 19–46.

Johnson, N.L., Kotz, S., Balakrishnan, N., 1995. Distributions in Statistics: Continuous Univariate Distributions, vol. 2, second ed. John Wiley, New York.

Kim, C.S., Phillips, P.C.B., 1999. Log periodogram regression: the nonstationary case. Cowles Foundation Discussion Paper, Yale University ⟨URL http://cowles.econ.yale.edu⟩.

Komlós, J., Major, P., Tusnády, G., 1976. An approximation of partial sums of independent RVs and the sample DF. II. Zeitschrift für Wahrscheinlichkeitsheorie und verwandte Gebiete 34, 33–58.

Künsch, H.R., 1986. Discrimination between long range dependence and monotonic trends. Journal of Applied Probability 23, 1025–1030.

Major, P., 1976. The approximation of partial sums of independent r.v.'s. Zeitschrift für Wahrscheinlichkeitsheorie und verwandte Gebiete 35, 213–222.

Phillips, P.C.B., 1987. Towards a unified asymptotic theory for autoregression. Biometrika 74, 535–547.

Phillips, P.C.B., 1999. Discrete Fourier transforms of fractional processes. Cowles Foundation Discussion Paper, Yale University ⟨URL http://cowles.econ.yale.edu⟩.

Phillips, P.C.B., Magdalinos, T., 2006a. Limit theory for moderate deviations from a unit root. Journal of Econometrics, forthcoming.

Phillips, P.C.B., Magdalinos, T., 2006b. Limit theory for moderate deviations from a unit root under weak dependence. In: Phillips, G.D.A., Tzavalis, E. (Eds.), The Refinement of Econometric Estimation and Test Procedures: Finite Sample and Asymptotic Analysis. Cambridge University Press, Cambridge forthcoming.

Phillips, P.C.B., Shimotsu, K., 2004. Local Whittle estimation in nonstationary and unit root cases. Annals of Statistics 32, 656–692.

Phillips, P.C.B., Solo, V., 1992. Asymptotics for linear processes. Annals of Statistics 20, 971–1001.

Robinson, P.M., 1994a. Time series with strong dependence. In: Sims, C. (Ed.), Advances in Econometrics, vol. 1. Cambridge University Press, Cambridge, pp. 47–96.

Robinson, P.M., 1994b. Efficient tests of nonstationary hypotheses. Journal of the American Statistical Association 89, 1420–1437.

Robinson, P.M., 1995. Log periodogram regression of time series with long memory dependence. The Annals of Statistics 23, 1048–1072.

Shimotsu, K., Phillips, P.C.B., 2005. Exact local whittle estimation of fractional integration. Annals of Statistics 33, 1890–1933.

Tanaka, K., 1999. The nonstationary fractional unit root. Econometric Theory 15, 549–582.

Velasco, C., 1999a. Non-stationary log-periodogram regression. Journal of Econometrics 91, 325–371.

Velasco, C., 1999b. Gaussian semiparametric estimation of non-stationary time series. Journal of Time Series Analysis 20, 87–127.

Velasco, C., 2000. Non-Gaussian log-periodogram regression. Econometric Theory 16, 44–79.

Zaitsev, A.Y., 1998. Multidimensional version of the results of Komlós, Major and Tusnády for vectors with finite exponential moments. ESAIM: Probability and Statistics 2, 41–108.